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EXACT SOLUTION OF THE PLANE PROBLEM FOR A COMPOSITE PLANE WITH A CUT ACROSS THE BOUNDARY BETWEEN TWO MEDIA*

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An exact closed solution is obtained for the problem of stress concentration in a composite elastic plane near a straight cut orthogonal to the dividing line between two media and which cuts it in half. The solution is constructed using the scheme of /1/ for the factorization of a special matrix coefficient of a Riemann problem. This Riemann problem is obtained by reducing the system of singular integral equations with a fixed singularity, which corresponds to the given problem of elasticity theory. The matrix coefficient of the Riemann problem does not satisfy the restrictions of /2/, and therefore the method described in /2/ produces an essential singularity at infinity for the factorizing matrices. The application of the scheme of /1/, based on the apparatus of boundary-value Riemann problems, on Riemann surfaces of algebraic functions /3/ enabled the essential singularity at infinity to be neutralized (by inversion of the corresponding Abelian integral).

The solution of the problem is obtained in quadratures in a form suitable for numerical realization. Working formulas are given for the stress intensity factors. A numerical example is examined.

1. Statement of the problem and its reduction to a vector Riemann problem. Let E_1 and ν_1 be the modulus of elasticity and Poisson's ratio of the halfplane $\Pi_- = \{x < 0, |y| < \infty\}$, and E_2 and ν_2 the corresponding quantities for the halfplane $\Pi_+ = \{x > 0, |y| < \infty\}$. The domains Π_+, Π_- are completely connected ($|y| < \infty$),

$$\| u, v, \sigma_x, \tau_{xy} \|_{x=-0} = \| u, v, \sigma_x, \tau_{xy} \|_{x=+0} \tag{1.1}$$

There is a cut $I = \{|x| < \varepsilon, y = \pm 0\}$ along the line $y = 0$ with the load $-p(x)$ applied to its edges:

$$\sigma_y|_{y=\pm 0} = -p(x), \quad \tau_{xy}|_{y=\pm 0} = 0 \quad (|x| < \varepsilon) \tag{1.2}$$

We consider the plane stressed state and it is required to find the stress intensity factors.

We define the function

$$\varphi(x) = E(x)\left[\frac{\partial v}{\partial x}\Big|_{y=-0} - \frac{\partial v}{\partial x}\Big|_{y=+0}\right], \quad E(x) = \begin{cases} E_1, & x < 0 \\ E_2, & x > 0 \end{cases} \tag{1.3}$$

Applying the generalized scheme of the integral transform method /4/, we reduce problem (1.1)-(1.3) to a system of two singular integral equations for the functions $\varphi_1(x) = \varphi(\varepsilon x)$, $\varphi_2(x) = -\varphi(-\varepsilon x)$, with a fixed singularity at the point where the cut intersects the boundary between the media,

$$\int_0^1 J_+(x, \xi)\varphi_1(\xi)\,d\xi + \int_0^1 S_-(x, \xi)\varphi_2(\xi)\,d\xi = p_1(x) \tag{1.4}$$

$$\int_0^1 S_+(x, \xi)\varphi_1(\xi)\,d\xi + \int_0^1 J_-(x, \xi)\varphi_2(\xi)\,d\xi = p_2(x) \quad (0 < x < 1)$$

**Prikl.Matem.Mekhan.*, 55,4,662-671,1991 where

$$J_{\pm}(x,\xi) = \frac{1}{\pi}\left[\frac{1}{\xi - x} + \frac{p_1^{\pm}\xi^2 + p_2^{\pm}\xi x + p_3^{\pm}x^2}{(\xi + x)^3}\right]$$

$$S_{+}(x,\xi) = \frac{1}{\pi}\frac{q_1^{\pm}\xi + q_2^{\pm}x}{(\xi + x)^2}, \quad p_j(x) = -4p((-1)^{j-1}\varepsilon x)$$

$$p_1^{\pm} = \delta_0^{-1}[\pm\nu_0(\pm\nu_0 - 2\mu^{\pm} - 6\mu^{\mp}) - 3(1-\mu)^2]$$

$$p_2^{\pm} = 4\delta_0^{-1}[\pm\nu_0(\mp\nu_0 + 2\mu^{\pm}) \pm 3(1-\mu^2)]$$

$$p_3^{\pm} = \delta_0^{-1}[-(\nu_0 + \mu - 1)^2 \pm 4(1-\mu^2)]$$

$$q_1^{\pm} = 8\delta_0^{-1}\mu^{\mp}(\pm\nu_0 + 2\mu^{\mp}), \quad q_2^{\pm} = 8\delta_0^{-1}\mu^{\mp}(1+\mu)$$

$$\delta_0 = (1 + 3\mu + \nu_0)(3 + \mu - \nu_0), \ \nu_0 = \nu_1 - \mu\nu_2, \ \mu^{+} = 1,$$

$$\mu^{-} = \mu = E_1E_2^{-1}$$

The solution of system (1.4) is sought in the class of Hölder functions in the interval $(0, 1)$ which admit of integrable singularities at the ends and satisfy the closure condition for the cut

$$\mu\int_0^1 \varphi_1(\xi)\,d\xi - \int_0^1 \varphi_2(\xi)\,d\xi = 0 \tag{1.5}$$

We extend system (1.4) to a semi-infinite interval using the functions

$$\varphi_{j+}(x) = -4\sigma_y((-1)^{j-1}\varepsilon x, 0) \quad (j = 1, 2)$$

and introduce the functions

$$\Phi_j(s) = \int_0^1 \varphi_j(\xi)\xi^s\,d\xi, \quad \Phi_j^{+}(s) = \int_1^{\infty} \varphi_{j+}(\xi)\xi^s\,d\xi$$

$$P_j^{-}(s) = \int_0^1 p_j(x)x^s\,dx$$

The functions $\Phi_j^{-}(s)$ and $P_j^{-}(s)$ are analytic in the halfplanes $\operatorname{Re} s > -\delta$ $(0 < \delta < 1)$ and $\operatorname{Re} s > -1$, respectively, and $\Phi_j^{+}(s)$ is analytic for $\operatorname{Re} s < 0$. Let $L = L_\gamma^{-} \cup C_\gamma \cup L_\gamma^{+}$, $L_\gamma^{\pm} = \{t \in C\colon \operatorname{Re} t = 0, \ \operatorname{Im} t \gtrless |\gamma|\}$, $-\delta < \gamma < 0$, $C_\gamma = \{t \in C\colon |t| = |\gamma|, \operatorname{Re} t < 0\}$. The contour L divides the complex plane C into two domains D^{+} and D^{-} $(\ni 0)$. The positive direction on L is chosen so that D^{+} is on the left when we traverse the contour. Clearly, the vector

$$\Phi(s) = \begin{Vmatrix}\Phi_1^{\pm}(s)\\ \Phi_2^{\pm}(s)\end{Vmatrix}, \quad s \in D^{\pm}$$

is piecewise-analytic with discontinuities along L.

Apply a Mellin transform to system (1.4). The result is the vector Riemann problem

$$\Phi^{+}(t) = G(t)\Phi^{-}(t) + g^{-}(t), \ t \in L \tag{1.6}$$

$$G(s) = b(s)I + c(s)A(s)$$

$$I = \begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix}, \quad A(s) = \begin{Vmatrix}l(s) & m_{-}(s)\\ m_{+}(s) & -l(s)\end{Vmatrix}, \quad g^{-}(t) = -\begin{Vmatrix}P_1^{-}(t)\\ P_2^{-}(t)\end{Vmatrix}$$

$$b(s) = \operatorname{ctg}\pi s + (r_0 + r_1s^2)\operatorname{cosec}\pi s, \ c(s) = \operatorname{cosec}\pi s$$

$$l(s) = -r_2 + r_3s^2, \ m_{\pm}(s) = -q_2^{\pm} + q_3^{\pm}s$$

$$r_0 = \delta_0^{-1}(\nu_0 + \mu - 1)^2, \ r_1 = 2\delta_0^{-1}[(1-\mu)^2 - \nu_0^2], \ r_2 = 4\delta_0^{-1}(1-\mu^2)$$

$$r_3 = 4\delta_0^{-1}(1+\mu)(\nu_0 + 1 - \mu), \ q_3^{\pm} = \pm 8\delta_0^{-1}\mu^{\mp}(\nu_0 + \mu - 1)$$

2. ***Investigation of the matrix* $G(s)$.** Consider the function

$$f(s) = l^2(s) + m_{+}(s)m_{-}(s) = a_0s^4 - a_1s^2 + a_2$$

$$a_0 = 16\delta_0^{-2}(1+\mu)^2(\nu_0 + 1 - \mu)^2, \ a_2 = 16\delta_0^{-2}(1+\mu)^4$$

$$a_1 = 32\delta_0^{-2}[2\mu(\nu_0 + \mu - 1)^2 + (1-\mu)(1+\mu)^2(\nu_0 + 1 - \mu)]$$

Clearly, $a_0, a_2 > 0$. It is easy to show that for any $\mu > 0$ and $0 < \nu_j < 1/2$ $(j = 1, 2)$, we have $a_1 > 0$. Let

$$D = a_1^2 - 4a_0a_2,\ \nu_- = \{1 - \nu_2 + [(1 - \nu_2)^2 + 4\nu_1]^{1/2}\}/2$$
$$\nu_+ = \{\nu_1 - 1 + [(\nu_1 - 1)^2 + 4\nu_2]^{1/2}\}(2\nu_2)^{-1}\ (\nu_- \leqslant \nu_+)$$

If $0 < \mu < \nu_-$ or $\mu > \nu_+$, then $D < 0$ and $f(s)$ has four complex-conjugate roots: $\pm s_1, \pm \bar{s}_1$, where

$$|s_1| = (a_2/a_0)^{1/4},\ \arg s_1 = {}^1/_2 \operatorname{arctg}(|D|^{1/2}a_1^{-1}) \in (0, \pi/2)$$

If $\nu_- < \mu < \nu_+$, then $D > 0$ and $f(s)$ has real and different roots: $\pm s_1, \pm s_2$, where

$$s_1 = [(a_1 - D^{1/2})(2a_0)^{-1}]^{1/2},\ s_2 = [(a_1 + D^{1/2})(2a_0)^{-1}]^{1/2}$$

Finally, in the two exceptional cases $\mu = \nu_\pm$ we have $D = 0$ and the roots are multiple: $\pm s_1, \pm s_1$, where

$$s_1 = [a_1(2a_0)^{-1}]^{1/2}$$

First consider the case $D \neq 0$. Let

$$\lambda_1(s) = b(s) + c(s)f^{1/2}(s),\ \lambda_2(s) = b(s) - c(s)f^{1/2}(s) \tag{2.1}$$

($\lambda_{1,2}$ are the characteristic functions of the matrix $G(s)$ /2/). To fix the branch $f^{1/2}(s)$ for the case $D < 0$, draw a cut (Fig.1) that joins the branching points $\pm s_1, \pm \bar{s}_1$ and which passes through the point $s = \infty$. We stipulate that ($\theta_1 = \arg s_1$)

$$-2\pi + \theta_1 < \arg(s - s_1) < \theta_1,\ -\pi - \theta_1 < \arg(s + \bar{s}_1) < \pi - \theta_1$$
$$-\pi + \theta_1 < \arg(s + s_1) < \pi + \theta_1,\ -\theta_1 < \arg(s - \bar{s}_1) < 2\pi - \theta_1$$

Then, in particular, $f^{1/2}(t) > 0,\ f^{1/2}(it) > 0$. For $D > 0$, the cuts are made joining the points s_1 and s_2, $-s_1$ and $-s_2$ (Fig.2) so that $-\pi < \arg(s \pm s_j) < \pi\ (j = 1, 2)$. The selected branch has the following properties: $f^{1/2}(t) > 0\ (-s_1 < t < s_1)$ and $f^{1/2}(it) > 0\ (-\infty < t < \infty)$.

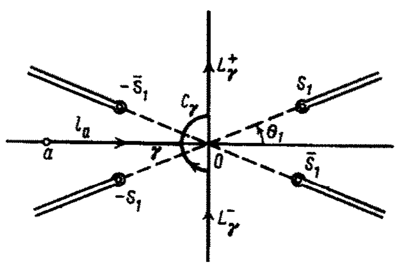

Fig.1

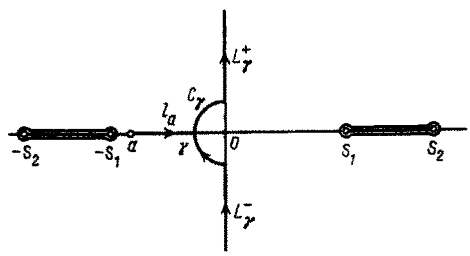

Fig.2

Analysis of the behaviour of the characteristic functions at zero and at infinity leads to the following results:

1) $\lambda_j(-0 + i\tau) \sim \mp i,\ \tau \to \pm\infty\ (j = 1, 2)$
2) $\lambda_1(\gamma) \sim \eta_1(\pi\gamma)^{-1},\ \lambda_2(\gamma) \sim -\eta_2\gamma\pi^{-1},\ \gamma \to -0$
   $\eta_1 = 8\delta_0^{-1}(1 + \mu)^2,\ \eta_2 = \pi^2/2 - r_1 - a_1(2a_2^{1/2})^{-1},\ \eta_j > 0$
3) $\operatorname{Im}\lambda_j(t) \leqslant 0,\ t \in L_\gamma^\pm\ (j = 1, 2)$

Hence $[\arg\lambda_1(t)]|_L = \pi,\ [\arg\lambda_2(t)]|_L = -\pi$ and therefore, as in /5/,

$$\varkappa_\Delta = \operatorname{ind}\{\lambda_1(t)\lambda_2(t)\} = 0,\ \varkappa_\varepsilon = \operatorname{ind}\{\lambda_1(t)[\lambda_2(t)]^{-1}\} = 1$$

Take the branches of the logarithms of the characteristic functions

$$-\pi/2 \leqslant \arg\lambda_j(t) \leqslant 3\pi/2,\ t \in L$$

Then, putting $\theta^{(j)}(t) = \arg\lambda_j(t)$, we obtain $(j = 1, 2)$:

$$\theta^{(j)}(t)|_{t\in L_\gamma^-} = \pi/2,\quad \theta^{(j)}(t)|_{t\in L_\gamma^+} = 3\pi/2 - 2\pi\delta_{j,2},\quad \theta^{(j)}(t)|_{t=\gamma} = \pi\delta_{j,1} \tag{2.2}$$

We have thus fixed the branch of the exponent $\varepsilon(t)$ of the matrix $G(t)$:

$$\varepsilon(t) = \frac{1}{2}\ln\left|\frac{\lambda_1(t)}{\lambda_2(t)}\right| + \frac{i}{2}\theta_\varepsilon(t),\quad \theta_\varepsilon(t) = \begin{cases} 0, & t \in L_\gamma^- \\ 2\pi, & t \in L_\gamma^+ \end{cases}$$
$$0 \leqslant \theta_\varepsilon(t) \leqslant 2\pi,\ t \in C_\gamma$$

To apply the factorization method of /2/, we need to have

$$E = \frac{1}{\pi i}\int_L \frac{e(t)}{f^{1/2}(t)}\,dt = 0$$

while in the case considered

$$E = g + ie \neq 0$$

$$g = \frac{1}{\pi}\int_0^\infty \ln\frac{\lambda_1(i\tau)}{\lambda_2(i\tau)}\frac{d\tau}{f^{1/2}(i\tau)} > 0, \quad e = \int_0^\infty \frac{d\tau}{f^{1/2}(i\tau)} > 0 \quad (\gamma = 0)$$

We will construct the factorizing matrices by the method described in /1/ using the boundary-value Riemann problem on a Riemann surface.

3. ***Factorization of the matrix*** $G(s)$. Take two copies C_1, C_2 of the extended complex s-plane $C \cup \{\infty\}$ with identically oriented cuts (Figs.1, 2) and glue the positive edges of the cuts in C_1 with to negative edges of the cuts in C_2 and the negative edges in C_1 to the positive edges in C_2. We obtain a two-sheeted Riemann surface R (of genus 1) /6/. The function $w(s)$ defined by the equation $w^2 = f(s)$ is single-valued on R and $w = f^{1/2}(s)$, $s \in C_1$, $w = -f^{1/2}(s)$, $s \in C_2$. Following /3/, we denote the point of the surface R with the affix $s = \alpha$ on the sheet C_1 by the pair $(\alpha, f^{1/2}(\alpha))$ and that on the sheet C_2 by the pair $(\alpha, -f^{1/2}(\alpha))$. The pair (s, w) uniquely defines a point on the surface R. We denote by ξ an analogue of the function w that satisfies the equation $\xi^2 = f(t)$. On the sheets C_1 and C_2, respectively, draw the contours L_1 and L_2, which are pointwise identical with L and have the same orientation. Define the contour Γ on the surface R by $\Gamma = L_1 \cup L_2$.

Represent the matrix $G(s)$ in the form

$$G(s) = \operatorname{ctg}\pi s G_0(s), \quad \operatorname{ctg}\pi s = K^+(s)[K^-(s)]^{-1} \tag{3.1}$$
$$K^+(s) = -\Gamma(-s)[\Gamma(1/2 - s)]^{-1}, \quad K^-(s) = \Gamma(1/2 + s)[\Gamma(1+s)]^{-1}$$

To factorize the matrix

$$G_0(t) = X_0^+(t)[X_0^-(t)]^{-1}, \quad t \in L \tag{3.2}$$

consider the boundary-value Riemann problem /3/ on the surface R:

$$F^+(t, \xi) = \lambda_0(t, \xi) F^-(t, \xi), \quad (t, \xi) \in \Gamma \tag{3.3}$$
$$\lambda_0(t, \xi) = \operatorname{tg}\pi t\,[b(t) + \xi c(t)] = 1 + (r_1 t^2 + r_0 + \xi)\sec\pi t \tag{3.4}$$

Let

$$\lambda_1^\circ(t) = \lambda_0(t, f^{1/2}(t)), \quad \lambda_2^\circ(t) = \lambda_0(t, -f^{1/2}(t)) \tag{3.5}$$

Then, comparing formulas (3.5) with (3.4), (2.1), we obtain from (2.2)

$$[\theta_1^\circ(t)]|_L = 0, \quad [\theta_2^\circ(t)]|_L = -2\pi, \quad (2\pi)^{-1}[\arg\lambda_0(t, \xi)]|_\Gamma = -1$$

where $\theta_j^\circ(t) = \arg\lambda_j^\circ(t)$ $(j = 1, 2)$ and $\theta_j^\circ(t)|_{t\in L_\gamma^\pm} = 0$. The function $\theta_1^\circ(t)$ is continuous on C_γ, and $\theta_2^\circ(t)$ varies from $\theta_2^\circ = \pi$ at the initial point $t = |\gamma|\exp\{i(\pi - 0)\}$ of the contour L to $\theta_2^\circ = 0$ at the point $t = i|\gamma|$. As the point t traverses the contour L_γ^+ and goes to L_γ^-, we have $\theta_2^\circ = 0$ up to the point $t = -|\gamma| i$. Then θ_2° diminishes to $\theta_2^\circ = -\pi$ at the point $t = \exp\{i(\pi + 0)\}$. Moreover,

$$\lambda_j^\circ(i\tau) \sim 1, \quad \tau \to \pm\infty; \quad \lambda_1^\circ(\gamma) \sim \eta_1, \quad \lambda_2^\circ(\gamma) \sim -\eta_2\gamma^2, \quad \gamma \to -0 \tag{3.6}$$

Following /3/, we write the solution of problem (3.3) in the form

$$F(s, w) = \exp\{\varphi(s, w)\} \tag{3.7}$$
$$\varphi(s, w) = \frac{1}{2\pi i}\int_\Gamma \ln\lambda_0(\tau, \xi)\frac{w + \xi}{2\xi}\frac{d\tau}{\tau - s} + \left(\int_{l_a} - n\int_{l_*}\right)\frac{w + \xi}{2\xi}\frac{d\tau}{\tau - s} \tag{3.8}$$

The real parameter a and the integer n are unknown.

The contours l_a and l_* lying on segments of the real axes of the two sheets of the surface R are defined for $D < 0$ and $D > 0$ in the following way:

1) $D < 0$. The contour l_* extends only on the first sheet from $-\infty$ to $+\infty$. The

contour l_a extends on the second sheet from the point (a, ξ_a) to the point (γ, ξ_γ), where $\xi_a = -f^{1/2}(a)$, $\xi_\gamma = -f^{1/2}(\gamma)$, and if $a > -|\gamma|$, then this contour passes through the point at infinity on the second sheet, always remaining on the real axis. In Fig.1, this contour is shown for the case $a < -|\gamma|$.

2) $D > 0$. The contour l_* extends on the first sheet from the point $(-s_1, 0)$ to $(s_1, 0)$ and on the second sheet from $(s_1, 0)$ back to $(-s_1, 0)$. The contour l_a traverses l_* $(l_a \subset l_*)$ in the opposite direction (relative to l_*) from the point (a, ξ_a) $(\xi_a = \delta f^{1/2}(a))$ to the point (γ, ξ_γ) $(\xi_\gamma = -f^{1/2}(\gamma))$ and the point (a, ξ_a) may lie both on the first sheet $(\delta = 1)$ and on the second sheet $(\delta = -1)$. The parameter δ, as well as a and n, will be defined below. In Fig.2, the contour l_a is shown for the case $\delta = -1$, $a < -|\gamma|$.

The function $F(s, w)$ does not have essential singularities (at infinity) if and only if

$$\frac{1}{2\pi i}\int_L \ln\frac{\lambda_1^\circ(\tau)}{\lambda_2^\circ(\tau)}\frac{d\tau}{f^{1/2}(\tau)} - n\int_{l_*}\frac{d\tau}{\xi} + \int_{l_a}\frac{d\tau}{\xi} = 0 \tag{3.9}$$

We have thus obtained the problem of the inversion of an elliptical integral of the first kind, which uniquely defines the parameters a, n and δ. The solution of problem (3.9) is given for the case $\gamma = -0$, which will be needed below. For $D < 0$, we have

$$\begin{gathered}a = (a_2/a_0)^{1/4}[(1 - \operatorname{cn}(u))(1 + \operatorname{cn}(u))^{-1}]^{1/2}\operatorname{sgn}(g - (2n+1)K_0)\\ n = E((2K_0)^{-1}g), \quad u = 2(a_0a_2)^{1/4}(K_0 - |g - (2n+1)K_0|)\\ K_0 = (a_0a_2)^{-1/4}\mathbf{K}(k), \quad k = {}^1/_2(2 + a_1(a_0a_2)^{-1/2})^{1/2}\\ g = \frac{1}{\pi}\int_0^\infty \ln\frac{\lambda_1^\circ(i\tau)}{\lambda_2^\circ(i\tau)}\frac{d\tau}{f^{1/2}(i\tau)}\end{gathered} \tag{3.10}$$

where $E(b)$ is the integer part of the number b, $\operatorname{cn}(u)$ is the elliptical cosine and $\mathbf{K}(k)$ is the complete elliptical integral of the first kind. For $D > 0$,

$$\begin{gathered}a = s_1\operatorname{sn}(a_0^{1/2}s_2u_0)\operatorname{sgn}(g - 2(2n+1)K_0)\\ n = E(g(4K_0)^{-1}), \quad u_0 = K_0 - |K_0 - |g - (2n+1)2K_0||\\ \delta = \operatorname{sgn}(K_0 - |g - (2n+1)2K_0|)\end{gathered}$$

The quantity g is defined in (3.10) and $\operatorname{sn}(u)$ is the elliptical sine. The function $\varphi(s, w)$ (3.8) has discontinuities that are multiples of $2\pi i$ on l_a and l_*. However, the function $F(s, w)$ is analytic in the neighbourhood of the points of the contours l_a and l_*, with the exception of the singular points $((a, \xi_a), (\gamma, \xi_\gamma)$, and the points at infinity). The values of $F(s, w)$ for $(s, w) \in l_a, l_*$ are calculated by passing to the limit from the domain of analyticity of the corresponding Cauchy integrals using Sokhotskii's formulas. Analysis of the integral representation of $F(s, w)$ (3.7)-(3.9) $(\gamma < 0)$ in the neighbourhood of the singular points shows that the function $F(s, w)$ does not vanish anywhere on R and is bounded on R with the exception of the point (a, ξ_a), where it has a simple pole

$$F(s, w) = O((s - a)^{-1}), \quad (s, w) \to (a, \xi_a)$$

The canonical matrix of solutions /7/ of the homogeneous Riemann problem

$$\Phi_0^+(t) = G_0(t)\Phi_0^-(t), \quad t \in L \tag{3.11}$$

is the matrix $X_0(s)$ defined by the relationships /1/

$$\begin{gathered}X_0(s) = [F(s, w)B(s, w) + F(s, -w)B(s, -w)]R(s)\\ X_0^{-1}(s) = R^{-1}(s)[F^{-1}(s, w)B(s, w) + F^{-1}(s, -w)B(s, -w)]\\ B(s, w) = w^{-1}B_0(s, w), \quad 2B_0(s, w) = wI + A(s)\\ R(s) = \|\rho_a, \rho_a'(s - a)\|, \quad \det X_0(s) = F(s, w)F(s, -w)(s - a)\end{gathered} \tag{3.12}$$

where ρ_a is the non-zero column of the matrix $-B_0(a, -\xi_a)$ and ρ_a' is the column vector such that $\det\|\rho_a, \rho_a'\| = 1$. The columns ρ_a, ρ_a' always exist, because rank $B_0(s, w) = 1$.

In the neighbourhood of the singular point $s = a$, the matrix $X_0(s)$ is bounded because of the identity $B_0(s, w)B_0(s, -w) \equiv 0$. The determinant of the matrix $X_0(s)$ is bounded everywhere in a finite part of the plane, and at infinity it is of order 1. The orders of the columns of the matrix $X_0(s)$ at infinity are 0 and 1, and the partial indices of problem (3.11) are therefore $\varkappa_1 = 0$, $\varkappa_2 = -1$.

To specify additional formulas for the case $m_+(a) \neq 0$, we take the matrix $R(s)$ in the form

$$R(s) = \left\| \begin{matrix} [\xi_a - l(a)]/2 & 2(s-a)\, m_+^{-1}(a) \\ -m_+(a)/2 & 0 \end{matrix} \right\| \tag{3.13}$$

4. Solution of the vector Riemann problem. Substituting the representations for $G(t)$ (3.1) and (3.2) into the boundary condition (1.6), we obtain

$$[K^+(t)\, X_0^+(t)]^{-1}\, \Phi^+(t) - \Psi^+(t) = [K^-(t)\, X_0^-(t)]^{-1}\, \Phi^-(t) - \Psi^-(t) \tag{4.1}$$

$$\Psi(s) = \frac{1}{2\pi i} \int_L [X_0^+(t)]^{-1} \frac{g^-(t)}{K^+(t)} \frac{dt}{t-s}$$

Let us determine the behaviour of $X_0^{-1}(s)$ at infinity. First let $D < 0$. If $s \to \infty$ and $|\arg s| < \theta_1$, $|\arg s| < \pi - \theta_1$, then

$$w = f^{1/2}(s) \sim a_0^{1/2} s^2, \quad B(s, \pm w) \sim {}^1/_2 \operatorname{diag}(1 \pm a_0^{-1/2} r_3, 1 \mp a_0^{-1/2} r_3) \tag{4.2}$$

Using condition (3.9), we obtain from (3.8) ($\gamma \to -0$):

$$\varphi(s, w) = \mu_a^- - \pi n i \operatorname{sgn} \operatorname{Im} s + O(s^{-1}) \tag{4.3}$$

$$\varphi(s, -w) = -\mu_a^- + {}^1/_2 \pi i (\operatorname{sgn} a + 1) \operatorname{sgn} \operatorname{Im} s + O(s^{-1}), \quad s \to \infty$$

$$\mu_a^- = \frac{a_0^{1/2}}{2} \int_{l_a} \frac{\tau\, d\tau}{f^{1/2}(\tau)} = -\frac{1}{4} \ln \left| \frac{2(a_0 f(a))^{1/2} + 2a_0 a^2 - a_1}{2(a_0 a_2)^{1/2} - a_1} \right| \quad (\gamma = -0)$$

Then by (3.7) we have

$$F(s, w) \sim (-1)^n \exp(\mu_a^-), \quad F(s, -w) \sim -\operatorname{sgn} a \exp(-\mu_a^-), \ s \to \infty \tag{4.4}$$

Now let $D > 0$. For the chosen branch $w = f^{1/2}(s)$ we have $w \sim -a_0^{1/2} s^2$, $s \to \infty$. Now,

$$B(s, \pm w) \sim {}^1/_2 \operatorname{diag}(1 \mp a_0^{-1/2} r_3, 1 \pm a_0^{-1/2} r_3), \quad s \to \infty \tag{4.5}$$

$$\varphi(s, \pm w) = \pm \mu_a^+ + O(s^{-1}), \quad F(s, \pm w) \sim \exp(\pm \mu_a^+), \quad s \to \infty$$

$$\mu_a^+ = \frac{a_0^{1/2}}{2} \int_{l_a} \frac{\tau\, d\tau}{\xi} = \frac{1}{4} \ln \left| \frac{(2a_0 s_1^2 - a_1)^{\delta+1}}{[2(a_0 a_2)^{1/2} - a_1]\, [2(a_0 f(a))^{1/2} + 2a_0 a^2 - a_1]^{\delta}} \right|$$

The matrix $R^{-1}(s)$ (the inverse of (3.13)) behaves at infinity as follows:

$$R^{-1}(s) \sim \left\| \begin{matrix} 0 & -2m_+^{-1}(a) \\ 0 & 0 \end{matrix} \right\|, \quad s \to \infty \tag{4.6}$$

Now using the asymptotic equalities (4.2) and (4.4)-(4.6), we obtain from (3.12)

$$X_0^{-1}(s) \sim \left\| \begin{matrix} 0 & \rho \\ 0 & 0 \end{matrix} \right\|, \quad s \to \infty \ (\rho = \text{const} \neq 0) \tag{4.7}$$

Noting the behaviour at infinity of the vector functions $[K^\pm(s)]^{-1}\, \Phi^\pm(s) = O(1)$, $\Psi(s) = o(1)$, $s \to \infty$, $s \in D^\pm$ and also of the matrix $X_0^{-1}(s)$ (4.7), we apply Liouville's theorem to equality (4.1) and find that (4.1) defines a vector function which is analytic everywhere in the C plane and is equal to $\|C, 0\|^\perp$ (C is an arbitrary constant). Thus, the solution of the Riemann problem (1.6) has the form

$$\Phi^\pm(s) = K^\pm(s)\, X_0^\pm(s)\, [\Psi^\pm(s) + \|C, 0\|^\perp], \quad s \in D^\pm \tag{4.8}$$

The constant C is obtained from the condition

$$\mu \Phi_1^-(0) - \Phi_2^-(0) = 0 \tag{4.9}$$

which follows from the closure condition for the cut (1.5). To compute $\Phi_j^-(0)$, we need formulas for $F(0, \pm f^{1/2}(0))$. First let $D < 0$. As in /5/, put $s = 0$, in (3.8) and then pass to the limit $\gamma \to -0$. Using (3.6), we

$$\lambda_0(\tau, f^{1/2}(\tau)) \sim \eta_1, \quad \lambda_0(\tau, -f^{1/2}(\tau)) \sim -\eta_2 \tau^2, \quad \tau \to 0$$

and then

$$\lim_{|\gamma| \to 0} F(0, f^{1/2}(0)) = (-1)^n \eta_1^{-1/2} \exp\{V_-(a)\}$$

$$\lim_{|\gamma| \to 0} F(0, -f^{1/2}(0)) = |a|^{-1} \eta_2^{-1/2} \exp\{-V_-(a)\}$$

$$V_-(a)=\frac{1}{2}\int_0^{|a|}\frac{f^{1/2}(0)-f^{1/2}(\tau)}{\tau f^{1/2}(\tau)}\,d\tau=\frac{1}{4}\ln\left|\frac{2a_2}{a_2-{}^1/_2a_1a^2+(a_2f(a))^{1/2}}\right|$$

For $D>0$, we obtain the formulas

$$\lim_{|\gamma|\to 0}F(0,f^{1/2}(0))=(-1)^{n+1}\operatorname{sgn}a\,\eta_1^{-1/2}\exp\{V_+(a)\}$$

$$\lim_{|\gamma|\to 0}F(0,-f^{1/2}(0))=(-1)^n\,|a|^{-1}\,\eta_2^{-1/2}\exp\{-V_+(a)\}$$

$$V_+(a)=\frac{1}{4}\ln\left|\frac{2a_2\,[a_2-{}^1/_2a_1a^2+(a_2f(a))^{1/2}]^\delta}{(a_2-{}^1/_2a_1s_1^2)^{\delta+1}}\left(\frac{s_1}{a}\right)^{2(\delta+1)}\right|$$

Substituting the values of $\Phi_j^-(0)$ from (4.8) into (4.9), we obtain

$$C=\frac{(\gamma_{21}-\mu\chi_{11})\,\Psi_1^-(0)+(\chi_{22}-\mu\chi_{12})\,\Psi_2^-(0)}{\mu\chi_{11}-\chi_{21}},\qquad \Psi^-(0)=\begin{Vmatrix}\Psi_1^-(0)\\ \Psi_2^-(0)\end{Vmatrix}$$

$$\chi_{j1}={}^1/_2\,[(\xi_a-l(a))\chi_j^+-m_+(a)\,\chi_{3-j}^-],\quad \chi_{j2}=-2am_+^{-1}(a)\,\chi_j^+\ (j=1,2)$$

$$\chi_1^\pm={}^1/_2\,[(1\mp a_2^{-1/2}r_2)\,F(0,f^{1/2}(0))+(1\pm a_2^{-1/2}r_2)\,F(0,-f^{1/2}(0))$$

$$\chi_2^\pm={}^1/_2a_2^{-1/2}q_2^\pm\,[-F(0,f^{1/2}(0))+F(0,-f^{1/2}(0))]$$

Note that $\xi_a=-f^{1/2}(a)$ for $D<0$ and $\xi_a=\delta f^{1/2}(a)$ for $D>0$. If $p(x)$ is a polynomial of degree N,

$$p(x)=\sum_{j=0}^{N}p_jx^j \tag{4.10}$$

the components of the vector $\Psi^-(0)$ are computed in explicit form:

$$\Psi^-(0)=4\sum_{j=0}^{N}\frac{\Gamma({}^3/_2+j)}{(j+1)!}\,[X_0^+(-j-1)]^{-1}\,p_je^j\begin{Vmatrix}1\\(-1)^j\end{Vmatrix} \tag{4.11}$$

5. ***The stress intensity factors*** K_I^+, K_I^-. We define the factors

$$K_I^\pm=\lim_{x\to\pm e\pm 0}[2\pi(\pm x-e)]^{1/2}\sigma_y(x,\pm 0) \tag{5.1}$$

$$N_k=\lim_{\xi\to 1-0}(1-\xi)^{1/2}\varphi_k(\xi)$$

Then by Abel's theorem

$$\Phi_k^-(s)\sim N_k\pi^{1/2}s^{-1/2},\quad s\to\infty,\quad s\in D^- \tag{5.2}$$

Let us determine from (4.8) the behaviour at infinity of the vector function $\Phi^-(s)$. We have

$$X_0(s)=[F_0+s^{-1}F_1+O(s^{-2})]\,(R_1+sR_0),\quad s\to\infty \tag{5.3}$$

$$R_0=\begin{Vmatrix}0 & r_{12}\\ 0 & 0\end{Vmatrix},\quad R_1=\begin{Vmatrix}r_{11} & -ar_{12}\\ -r_{12}^{-1} & 0\end{Vmatrix},\quad F_0=\begin{Vmatrix}x_+ & 0\\ 0 & x_-\end{Vmatrix}$$

$$r_{11}={}^1/_2\,[\xi_a-l(a)],\quad r_{12}=2m_+^{-1}(a)$$

$$x_\pm=\begin{cases}{}^1/_2\,[(-1)^n\exp(\mu_a^-)\,A_0^\pm-\operatorname{sgn}a\exp(-\mu_a^-)\,A_0^\mp], & D<0\\ {}^1/_2\,[\exp(\mu_a^+)\,A_0^\mp+\exp(-\mu_a^+)\,A_0^\pm], & D>0\end{cases}$$

$$A_0^\pm=1\pm a_0^{-1/2}r_3$$

(F_1 is a (2×2) matrix, $\mu_a^\pm$ are defined by (4.5) and (4.3)). Substituting (5.3) into (4.8) and using (5.2), we obtain

$$\pi^{1/2}\left\|\begin{matrix}N_1\\N_2\end{matrix}\right\| = F_0R_0\left\|\begin{matrix}\Psi_1^\circ\\\Psi_2^\circ\end{matrix}\right\| + F_0R_1\left\|\begin{matrix}C\\0\end{matrix}\right\|$$

$$\Psi^\circ = -\frac{1}{2\pi i}\int [X_0^+(t)]^{-1}\frac{g^-(t)}{K^+(t)}\,dt, \quad \Psi^\circ = \left\|\begin{matrix}\Psi_1^\circ\\\Psi_2^\circ\end{matrix}\right\|$$

Using the asymptotic equalities

$$\sigma_y(\varepsilon x, \pm 0) \sim -\frac{1}{4\pi}\int_0^1 \frac{\varphi_j(\xi)}{\xi - x}\,d\xi, \quad x \to 1+0, \quad j = 1, 2$$

$$\int_0^1 \frac{(1-\xi)^{-1/2}}{\xi - x}\,d\xi \sim -\pi(x-1)^{-1/2}, \quad x \to 1+0$$

we find the coefficients (5.1):

$$K_I^+ = \varepsilon^{1/2}2^{-3/2}x_+(Cr_{11} + \Psi_2^\circ r_{12}), \quad K_I^- = -\varepsilon^{-1/2}2^{-3/2}x^- C r_{12}^{-1}$$

For a polynomial load (4.10), we have

$$\Psi^\circ = 4\sum_{j=0}^{N}[X_0^+(-j-1)]^{-1}\left\|\begin{matrix}1\\(-1)^j\end{matrix}\right\| p_j\varepsilon^j\frac{\Gamma(3/2+j)}{j!} \tag{5.4}$$

We will give the formulas $(D<0)$ for $F(x, \pm f^{1/2}(x))$ (x is a real number), which are needed in order to evaluate the values of the matrix $[X_0^+(-j-1)]^{-1}$ occurring in (4.11) and (5.4):

$$F(x, f^{1/2}(x)) = (-1)^n R_1^+R_2, \quad F(x, -f^{1/2}(x)) = x(a-x)^{-1}\operatorname{sgn} a\, R_1^-R_2^{-1}$$

$$R_1^\pm = \exp\left\{-\frac{x}{2\pi}\int_0^\infty\left[\ln(\lambda_1^\circ(it)\lambda_2^\circ(it)) \pm \left(\frac{f(x)}{f(it)}\right)^{1/2}\ln\frac{\lambda_1^\circ(it)}{\lambda_2^\circ(it)}\right]\frac{dt}{t^2+x^2}\right\}$$

$$R_2 = \exp\{w_a(x) + [n + 1/2(\operatorname{sgn} a + 1)]\,w^\circ(x)\}$$

$$w_a(x) = \frac{1}{2}\int_0^{|a|}\frac{f^{1/2}(x) - f^{1/2}(\tau)}{f^{1/2}(\tau)}\frac{d\tau}{\tau - x\operatorname{sgn} a}$$

$$w^\circ(x) = x\int_0^\infty\frac{f^{1/2}(x) - f^{1/2}(\tau)}{f^{1/2}(\tau)}\frac{d\tau}{x^2 - \tau^2}$$

Note that for $D=0$ the solution of the problem can be obtained both by passing to the limit as $D \to \pm 0$ and directly /1/ (the genus of the surface R is zero). Morevoer, the corresponding Riemann problem can be solved in this case by following /8, 5/.

As a numerical example, consider the case when $p(x) = p = \text{const}$, $\varepsilon = 1$. Table 1 shows $p^{-1}K_I^+$ and $p^{-1}K_I^-$ as functions of $\mu = E_2^{-1}E_1$ for selected ν_1, ν_2. As $\mu \to 1$ ($\nu_1 = \nu_2$), K_I^+ and K_I^- tend to $K_I = \pi^{1/2}p$, which is the value of the stress intensity factor for normal separation of a homogeneous plane.

Table 1

μ	$\nu_1 = \nu_2 = 0.3$		$\nu_1 = 0.2,\ \nu_2 = 0.3$	
	$p^{-1}K_I^+$	$p^{-1}K_I^-$	$p^{-1}K_I^+$	$p^{-1}K_I^-$
0.02	2.75	1.13	2.77	1.13
0.1	2.48	1.27	2.45	1.27
0.2	2.27	1.38	2.27	1.39
0.4	2.04	1.53	2.03	1.53
0.6	1.91	1.62	1.90	1.63
0.8	1.82	1.70	1.82	1.70
1.0	1.77	1.77	1.76	1.77

For all the values of μ, ν_1, and ν_2 used in the numerical calculations, the parameter a is negative, $n=0$, and for $D>0$ we have $\delta = -1$ (the point (a, ξ_a) lies on the second sheet of the surface R). Since $D>0$ for $\nu_- < \mu < \nu_+$, and ν_-, ν_+ are close to each other (for $\nu_1 = 0.2$, $\nu_2 = 0.3$, for instance, $\nu_- \approx 0.9179$ and $\nu_+ \approx 0.9274$), the case $D<0$ is more interesting for numerical implementation.

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EQUILIBRIUM OF A SYSTEM OF CRACKS WITH CONTACT AND OPENING REGIONS*

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The equilibrium of a system of rectilinear cracks is considered within the framework of the plane theory of elasticity, taking into account the possibility of the formation of contact regions on their surfaces. In this case a jump in the normal displacement is specified on a part of the crack surface (within the area of contact), and a normal stress in the opening region. The shear stress is specified along the whole crack.

The well-known integral equations (IE) obtained for cracks without contact regions /1-3/ cannot be used to solve the problem in question, since the loads in them are assumed given along the whole crack, whereas within the regions of contact between its surfaces the normal stresses are not known.

In order to overcome this difficulty, a different method of deriving the IE is proposed, describing the distribution of the jump in displacement along the crack. The possibility of representing the solution of the initial problem in the form of the sum of solutions of two problems for the initial crack, namely, of the problem of a crack with an unknown jump in shear displacements, but with shearing loads specified along it, and of the problem of determining the opening regions along the initial crack with unknown normal displacement jump and normal loads specified in these regions, is used. This makes it possible to obtain a system of IE written for the contact and opening regions, respectively, with separated right-hand sides. In one equation the right-hand side contains the known normal stresses, and in the other the shear stresses. The solution of the resulting system yields the distribution of displacement jumps along the crack and the unknown boundaries of the contact and opening regions. The condition determining the position of the opening regions is that there are no singularities in the stress distribution near the unknown boundaries of these regions /4/.

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